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Towards a formal theory of development/evolution and characterization of time discretized operators for heat transfer

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Abstract *The time-discretization process of transient equation systems is an important concern in computational heat transfer applications. As such, the present paper describes a formal basis towards providing the theoretical concepts, evolution and development, and characterization of a wide class of time discretized operators for transient heat transfer computations. Therein, emanating from a common family tree and explained via a generalized time weighted philosophy, the paper addresses the development and evolution of time integral operators [IO], and leading to integration operators [InO] in time encompassing single-step integration operators [SSInO], multi-step integration operators [MSInO], and a class of finite element in time integration operators [FETInO] including the relationships and the resulting consequences. Also depicted are those termed as discrete numerically assigned [DNA] algorithmic markers essentially comprising of both: the weighted time fields, and the corresponding conditions imposed upon the dependent variable approximation, to uniquely characterize a wide class of transient algorithms. Thereby, providing a plausible standardized formal ideology when referring to and/or relating time discretized operators applicable to transient heat transfer computations.*

Introduction

The development of computational algorithms for parabolic transient systems of equations have matured over the years. Analytical approaches, although indispensable are not economically feasible and/or impractical for complex linear/nonlinear situations. This is especially true for large scale engineering

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computations and thus provides the impetus and the need for developing effective computational algorithms for transient analysis. After the semi-discretization of transient field problems which in the case of linear problems leads to a set of ordinary differential equations in time, there exist many numerical approximation methods which have been introduced for the time discretization and the solution of these classes of problems. These include finite difference approximations for the time derivatives which lead to the so called direct time integration relevant one-step and multi-step methods, attempts towards providing unified formulations via a weighted residual approach, and alternate viewpoints and insights describing the underlying theoretical basis for characterizing time discretization operators via a generalized time weighted philosophy (see Belytschko and Hughes (1983); Wood (1987, 1990); Zienkiewicz and Taylor (1994), and Tamma *et al.* (1997)); mode superposition type methods (Bathe, 1982); finite element formulations in space and time (Oden, 1969; Fried, 1969; Argyris and Scharpf, 1969); alternate approaches employing variational principles in time which also lead to similar forms of algorithms as in the weighted residual approach (Washizu, 1975; Gurtin, 1964); hybrid formulations which employ transform methods (Laplace/ Fourier) in conjunction with the standard Galerkin procedures and space finite elements and then numerically invert the resulting representations to obtain the solutions at desired times of interest (Manolis and Beskos, 1980; Tamma and Railkar, 1987a, 1987b); and the like.

Of the various computational algorithms available in the literature for transient field problems, the so called direct time integration approaches have been consistently popular and most common in production codes. In other related efforts, recent advances in the theoretical development of transient algorithms encompassing modal based time integral operators, time integration operators and the like including a plausible theory of development/ evolution of computational algorithms are described by Tamma *et al.* (1997). Also touched upon are the theoretical developments towards bridging the relationships between those termed as integral and integration time operators (details are described subsequently) via a generalized time weighted philosophy with a clear insight and specific knowledge of the weighted time fields (not specifically known previously) which now permits for the first time a basis for providing the underlying distinction between integral and integration operators in time. An overview of the general developments for parabolic and hyperbolic-parabolic transient systems is recently described in Tamma *et al.* (1997).

Specifically focusing attention on the various approaches outlined above, the following general consensus and inferences can be briefly drawn. Direct time-integration methods for transient field problems have long been a subject matter of widespread research activity. To date, much progress has been made in the development and understanding of the direct time-integration methods. This includes the development of alternate theoretical formulations which are different from the original methods of algorithmic development, thereby leading to efficient algorithmic representations and interpretations, attempts towards unified formulations, investigations encompassing accuracy and stability properties, formulations of variable and mixed time integration approaches, adaptive time stepping approaches, implementation aspects and the like. These direct time integration algorithms are generally categorized as explicit methods and implicit methods, and are subsequently addressed. Whereas for linear situations, long time durations, and inertial type problems, traditional mode superposition type approaches and practices are attractive, thus far, they have not been as popular as direct time-integration techniques for non-linear situations (because of cited reasons in the literature which indicate the need to frequently compute the associated eigen problems repeatedly to satisfy local mode superposition). However, there is a general consensus that modal analysis is felt to be more efficient if many analyses of the same configuration are necessary, for long time durations, and/or if only a small number of modes dominate the solution. On the other hand, direct time integration techniques continue to be popular in commercial codes.

To address some of the disadvantages of direct time-stepping approaches such as the widely advocated α-family of methods, and to overcome some of the deficiencies and the practical inability of traditional modal analysis methods for general transient nonlinear situations, Tamma *et al.* (1994, 1995), Mei *et al.* (1994) and Chen *et al.* (1993) have described via a generalized time weighted philosophy the theoretical basis for formulating the exact solution and consequences leading to an explicit, unconditionally stable VIrtual-Pulse (VIP) time integral method of computation. Although the methodology emanates via a time weighted philosophy, it is based on new and different perspectives in an attempt to capitalize not only on the advantages of both, but also towards providing a fundamentally sound theoretical basis and avenues for describing time integral operators. Therein, the fundamental developments towards establishing the theoretical basis for subsequent applications to linear and a class of nonlinear transient heat transfer analysis influenced by conduction, convection, and radiation heat transfer mechanisms are described and the pros and cons of such approaches for practical problems are identified. Via the time integral methodology, after the so-called semi-discretization process, the time discretization is achieved via a virtual or weighted time field with the time weighting fields proposed being uniquely selected so to account for the physics of the problem; thereby, resulting in an explicit time integral methodology whose integral operator naturally inherits certain computationally attractive features and good stability/accuracy attributes. The pros and cons of such time integral operators, however, need to be understood by the analyst and have been described.

Formulations employing finite elements in space and time have indeed received some attention since the pioneering efforts in 1969. However, although effective for a class of situations, they continue to face certain difficulties as related to the size of the resulting formulations, storage, and the like in comparison to traditional direct time stepping formulations. There also exist other relevant, more recent computational algorithms and approaches to effectively tackle transient field problems. The appropriate technique depends heavily on the problem under consideration.

Of particular interest in this paper are the class of transient field problems, which, as a result of the semi-discretization process (in a finite element sense), yield the following simultaneous ordinary differential equations which can be represented in matrix form for linear problems as:

$$\mathbf{C}\dot{\vec{T}} + \mathbf{K}\vec{T} = \vec{F} \tag{1}$$

with the initial condition

$$\vec{T}(0) = \vec{T}_0 \tag{2}$$

Furthermore, in many practical engineering situations, nonlinearities exist thus altering the above equations to imply,

$$\mathbf{C} = \mathbf{C}\left(\vec{T}\right), \mathbf{K}\vec{T} = \mathbf{P}\left(\vec{T}\right), \vec{F} = \vec{F}\left(\vec{T}\right)$$

where $\mathbf{C}$ is the capacitance matrix, $\mathbf{K}$ is the conductance matrix which is comprised of the effects due to conduction, convection and radiation, and $\vec{F}$ is the corresponding load vector which also includes internal heat generation. The primary motivation and objectives of this paper follow next. Currently, there exist the original methods of development of various computational algorithms in the literature and other previous efforts such as the weighted residual approach and the like describing alternate formulations which include these original developments and have indeed provided certain useful generalizations. However, these previous efforts fail to enact a mathematically consistent formulation in developing these generalizations. Thus, leading to not only a clear lack of recognition of the underlying theoretical relevance and insight and burden carried by the weighted time fields and the corresponding conditions imposed upon the associated dependent field variable approximations in the course of the subsequent developments, but also hampering the systematic bridging of the relationships amongst time discretized operators encompassing those termed as integral and integration operators in time including their characterization. This is the primary focus of the present manuscript. Unlike past efforts for the development of computational algorithms for transient analysis, the theoretical basis and framework of the present developments, although they emanate from a virtual field or a weighted time field introduced for enacting the time discretization, we herein seek to first formulate in a mathematically consistent manner an equivalent representation containing the adjoint operator associated with the original transient semi-discretized equation system. This equivalent representation forms the theoretical backbone and is formulated via a consistent integration by parts (once, twice, etc., based on the order of the time derivatives appearing in the semi-discretized system) and is first capitalized upon instead of directly dealing with the original time weighted semi-discretized equation system. Consequently, it therein serves as a prelude towards a clear understanding and an improved insight and provides a basis towards a formal theory of development, evolution and characterization of time discretized operators within a general framework including bridging of the relationships for both integral/integration time operators. Subsequently, the selection or the burden carried by the virtual or weighted time field originally introduced for enacting the time discretization process determines the formal outcome of "exact integral operators", "approximate integral operators", and a wide class of "integration operators" including identifying the underlying basis for the conditions imposed on the selection of the corresponding dependent field variable approximations. The "burden of weight", so phrased due to the presence and role of the virtual or weighted time field introduced for enacting the time discretization, and the underlying conditions imposed upon the corresponding approximation of the dependent field variable, both play a fundamental part in subsequently enabling a formal basis for characterization of computational algorithms via discrete numerically assigned [DNA] algorithmic markers which are associated with both the above.

The paper is arranged as follows. The theoretical basis and evolution and classification of time discretized operators starting from the formal development of time integral operators and the resulting consequences systematically leading to time integration operators is first established. Subsequently, the characterization of computational algorithms via discrete numerically assigned [DNA] algorithmic markers for a generalized family of integral/integration operators is described. That which is pertinent to characterization are the so called DNA algorithmic markers which comprise of both the weighted time fields and the corresponding imposed conditions on the dependent field variable approximation. These issues are particularly addressed. A wide class of known and established methods are then shown to be subsets of the family of single-step integration operators [SSInO] which can be uniquely characterized. Next, the development of multi-step integration operators [MSInO] from the development of these single-step time operators is described and the relevant issues and equivalence are fundamentally explained and identified. Finally, the relationships between a class of so-called finite element in time integration operators [FETInO] and the multi-step representations are established.

Time integral/integration operators for transient analysis

The primary motivation and objectives here are the theoretical ideas leading to the development of a formally standardized family of time integral/integration operators for the solution of transient field problems. In the $\mathbf{W}_p$-family of time integral/integration operators, $p = 0, 1, 2, 3\ldots$ denote the various classes of time operators resulting from the theoretical developments described subsequently for transient field problems. We designate $p = 0$, namely, $\mathbf{W}_0 = \mathbf{W}$ as integral operators in time, and the consequences systematically leading to $p = 1, 2, 3, \ldots$, namely, W_p as time integration operators. Subsequently, in later sections, further developments leading to multi-step time operators and a class of finite element in time operators and their associations are briefly described.

Theoretical basis: formal theory of formulation of algorithms, and development/evolution

It is herein postulated that time discretized operators encompassing both time integral and a wide class of time integration operators pertain to and emanate from the same family, with the burden carried by a virtual field or weighted time field specifically introduced for enacting the time discretization. The subsequent developments are strictly enacted in a mathematically consistent manner so as to first permit obtaining the adjoint operator associated with the original semi-discretized system. Consequently, this serves as a prelude for the classification of time discretized operators and therein permits the characterization of computational algorithms for transient analysis as described next.

Consider the semi-discretized form of the equation system obtained for linear transient field problems (say in a finite element sense) following the usual space discretization procedures:

$$\begin{aligned} C\dot{\vec{T}} + \mathbf{K}\vec{T} &= \vec{F} \\ \vec{T}(0) &= \vec{T}_0 \end{aligned} \tag{3}$$

Assuming an arbitrary virtual field or weighted time field, $\mathbf{W}(t)$, for enacting the time discretization process, the above semi-discretized equation system can be cast into the form:

$$\int_{t_n}^{t_{n+1}} \mathbf{W}^T \left(\mathbf{C}\dot{\vec{T}} + \mathbf{K}\vec{T} - \vec{F} \right) dt = 0 \tag{4}$$

Following (Tamma *et al.*, 1994; 1995, 1997; Chen *et al.*, 1993; Mei *et al.*, 1994) we propose to first reduce in a mathematically consistent manner the above equation system (4) so as to first yield the adjoint operator associated with the original semi-discretized transient field system. This forms the theoretical framework and basis for enabling the formal development of time integral operators and subsequently bridging the relationship between integral and integration time operators. Accordingly, for the above first-order system we integrate by parts once the term containing the time derivative of the temperature vector. Thus, we have

$$\int_{t_n}^{t_{n+1}} (\dot{\mathbf{W}}^T\mathbf{C} - \mathbf{W}^T\mathbf{K})\vec{T}dt = \mathbf{W}^T\mathbf{C}\vec{T}\,\Big|_{t_n}^{t_{n+1}} - \int_{t_n}^{t_{n+1}} \mathbf{W}^T\vec{F}dt \tag{5}$$

where we define:

$$\mathbf{W}_{Adj} \equiv \dot{\mathbf{W}}^T C - \mathbf{W}^T \mathbf{K} \quad (6)$$

At this juncture, the primary focus is on the fundamental theory leading to a clear insight into the role of the weighted time fields. With the exception that $\mathbf{W}$ and consequently $\mathbf{W}_{Adj}$ is undefined, thus far, there are no approximations introduced in formulating the reduced equivalent representative form containing the adjoint operator associated with the original transient semi-discretized system. The burden of weight carried by $\mathbf{W}$ originally introduced to enact the time discretization process and now related to the adjoint operator, $\mathbf{W}_{Adj}$, and the resulting consequences in the evolution of $\mathbf{W}$ encompass:

- the theoretical (exact) solution, namely $\mathbf{W}_{Exact}$, which is a matrix representation, is obtained by setting the adjoint operator equal to zero with/without considerations involving the notion of introducing transformation into modal basis (i.e. associating an eigen problem in the development of the time discretized operator);
- approximations further enacted upon the theoretical form of $\mathbf{W}_{Exact}$ leading to $\mathbf{W}_{Approx}$ which, however, still preserves the matrix representations; and
- further approximations (which can theoretically explain the underlying reasons) leading to a degenerated representation of the exact form of the theoretical weighted time fields as instead a vector $\vec{W}(t)$ or single valued scalar function of time, $W(t)$, which do not preserve the original matrix form.

The aforementioned choices thereby describe (based on the assumptions invoked) the development of "exact integral operators", "approximate integral operators", or a wide class of other "integration operators" for transient field problems. Furthermore, these choices of the weighted time fields also additionally impose specific conditions on the corresponding approximations for the dependent field variable. As a consequence, both of the above govern the characterization of time discretized operators and serve as a prelude towards formally establishing discrete numerically assigned [DNA] algorithmic markers to uniquely characterize a wide class of computational algorithms.

Summarizing and following Tamma *et al.* (1997, 1998, 1999), the formal theoretical developments and the evolution of time discretized operators emanating from a generalized time weighted philosophy for enacting the time discretization process is presented and the following is noteworthy. The proposition of a mathematically consistent formulation of the semi-discretized transient system in a time weighted sense which first leads to an equivalent representation containing the adjoint operator associated with the original semi-discretized transient system is shown to provide a basis for the general classification of time discretized operators. The resulting Type 1, Type 2, and Type 3 classifications pertain to time discretized operators, wherein, the burden is placed upon the choices for the weighted time fields and the above classifications are indeed a direct consequence, all evolving from a theoretically exact time weighted representation. In principle, a plausible theory of evolution of time discretized operators is described for a wide class of computational algorithms emanating from the exact solution and explained via a generalized time weighted philosophy. Thus, following Tamma *et al.* (1998; 1999), Type 1 operators in time are cited to be a direct result of a mathematically consistent choice of the weighted time fields which are theoretically exact, $\mathbf{W}_{Exact}$, and obtained by setting the adjoint operator equal to zero, thereby leading to those termed as time integral operators. For this selection of the weighted time fields, there is no need to impose conditions upon the dependent field variable and is irrelevant as evident from the equivalent representation containing the adjoint operator, equation (5). Type 2 operators are cited to be those time operators resulting from introducing approximations to these exact weighted time fields leading to $\mathbf{W}_{Approx}$, which however, still preserve the original matrix representation of the theoretically exact weighted time fields. It is important to note that in both Type 1 and Type 2 classifications, there is no need to impose conditions on the corresponding approximations for the dependent field variable and is irrelevant as discussed earlier. On the other hand, attempts leading to a degenerated representation (in a theoretical sense) of the matrix form of the theoretical weighted time fields leading to other forms of approximation (such as a single valued scalar function of time) which do not preserve the theoretical form are shown to be able to explain and lead instead to the classification of Type 3 time operators such as the family of single-step integration operators in time [SSInO] of which several of the widely advocated and so-called time integration schemes are subsets of this family. Here, a corresponding consistent approximation of the dependent field variable is however important and must be invoked. As a consequence, the following inferences can be postulated and/or further drawn:

- that there exists, emanating from a common family tree, and explained via a generalized time weighted philosophy, a wide class of time discretized operators all evolving from the theoretical (exact) solution and termed here as the generalizations of a standardized family which can be classified as Type 1, Type 2, and Type 3 time operators, and uniquely characterized by discrete numerically assigned [DNA] algorithmic markers which comprise of both the weighted time fields and the corresponding conditions imposed on the approximations for the dependent field variable;
- from the formal developments for "exact integral operators", the resulting "approximate integral operators" and a wide class of time "integration operators and known methods" are simply subsets which can be uniquely characterized; and
- different from the way we have been looking in the past at developments encompassing modal type and a wide class of time stepping approaches,

and significantly different from the way these have been developed and described in standard text books over the years, all of these are indeed associated, and emanate from the same family tree with common roots.

Development of time integral operators [IO]

Via a generalized time weighted philosophy for the development of time discretized operators, we first consider the development of integral operators in time. This can be accomplished both with and without considering the notion of introducing transformation to modal basis (that is introducing modal transformation and requiring an eigen problem to be associated in the development of the time integral operators). And, the subsequent developments are based on the unique selection of the weighted time fields as the theoretical (exact) solution obtained by setting equal to zero the adjoint operator contained in the equivalent representative form of the original semi-discretized transient system.

In view of the above considerations, following Tamma *et al.* (1997), we may consider the selection of $\mathbf{W}$ as either the solution of $\mathbf{W}_{Adj} = \dot{\mathbf{W}}^T\mathbf{C} - \mathbf{W}^T\mathbf{K} = 0$ or $\mathbf{W}_{Adj} = \dot{\mathbf{W}}^T\mathbf{C} - \mathbf{W}^T\mathbf{K} = \mathbf{W}^T(\tau)\mathbf{C}\delta(t-\tau)$ where $\mathbf{W}(\tau) = \mathbf{I}$, which is the associated integration factor, or equivalently, the Greens function respectively. A theoretically exact time integral operator may now be readily derived for either of the above selections and with or without introducing the notion of transformation to modal basis. Both the resulting representations pertain to Type 1 classification of time operators where the resulting $\mathbf{W}$ are the theoretical weighted time fields consistently obtained by setting the adjoint operator, $\mathbf{W}_{Adj} = 0$ as described above (for details see Tamma *et al.*, 1997). That involving the notion of introducing transformation to modal basis leading to such Type 1 time integral operators is briefly detailed next and subsequently shown to systematically lead to the formal development of Type 3 classification of time integration operators and their characterization (which is the focus of the present manuscript) via a degenerated form of representation of the theoretical weighted time fields. The latter, namely, without involving the notion of modal transformation but introducing approximations for the weighted time fields which still preserves the original theoretical form is recently described elsewhere (Tamma *et al.*, 1999) leading to new attractive algorithms as viable alternatives and termed as Type 2 classification of time operators.

For the purposes of demonstrating the theoretical developments, focusing attention on the notion of introducing transformation to modal basis following Tamma *et al.* (1997), we have

$$\vec{T} = \mathbf{X}\vec{\Theta} \tag{7}$$

and

$$\mathbf{W} = \mathbf{X}\mathbf{W}_{\Theta} \tag{8}$$

where $\mathbf{X} = \left[\vec{X}_1\ \vec{X}_2\ \ldots\ \vec{X}_n\right]$ denotes the matrix of eigenvectors and $\mathbf{W}_{\Theta}$ are obtained as described subsequently.

Consider next the solution of the eigenproblem related to the transient field problem based on the initial state at $t = t_n$:

$$\mathbf{KX} = \mathbf{CX}\Omega \tag{9}$$

where $\Omega = diag[\lambda_1, \lambda_2, \ldots, \lambda_n]$ is the matrix associated with the eigenvalues, and $\mathbf{X}$ is the corresponding matrix of eigenvectors.

Introducing equation (8) into either selection of $\mathbf{W}_{Adj} = 0$ as described earlier with the appropriate initial condition, results in the theoretically exact matrix form of the weighted time fields (the latter option for $\mathbf{W}_{Adj} = 0$ is shown here) as

$$\mathbf{W}(t) = \mathbf{X}\mathbf{W}_{\Theta_\tau} e^{\Omega(t-\tau)} H(t-\tau) \tag{10}$$

where $H(t-\tau)$ is the Heaviside function, $e^{\Omega} = diag\left[e^{\lambda_1}, e^{\lambda_2}, \cdots, e^{\lambda_n}\right]$, and $\mathbf{W}_{\Theta_\tau} = \mathbf{W}_{\Theta}(t = \tau)$ and where we define

$$\mathbf{W}_{\Theta}(t) \equiv \mathbf{W}_{\Theta_\tau} e^{\Omega(t-\tau)} H(t-\tau) \tag{11}$$

Introducing the modal transformations for $\vec{T}$ and $\mathbf{W}$ from equation (7) and equation (8) into equation (5) and doing the algebra, results in the "exact integral operator" in time which is the theoretical solution and is given as

$$\vec{T}_{n+1} = \mathbf{C}^{-1}\mathbf{X}^{-T} e^{-\Omega\Delta t}\mathbf{X}^T\mathbf{C}\vec{T}_n + \mathbf{C}^{-1}\mathbf{X}^{-T} e^{-\Omega\Delta t} \int_{t_n}^{t_{n+1}} e^{\Omega(t-t_n)}\mathbf{X}^T\vec{F}dt \tag{12}$$

The "exact integral operator" thus derived and its consequences leading to an "approximate integral operator" and termed as the VIrtual-Pulse (VIP) time integral methodology is described in Tamma *et al.* (1997) and references thereof. The "approximate integral operator" is constructed by making an approximation to the forcing function, $\vec{F}$. For transient field problems, this first leads to an explicit self-starting time integral methodology of computation which naturally inherits excellent algorithmic stability, and certain attractive computational and accuracy attributes. It is of n-th order accuracy for (n-1)th order approximation of the load $\vec{F}$. The comparative stability characteristics of the trapezoidal family are shown in Figure 1 and Table V with the explicit second-order accurate approximate integral operator in time.

Nonlinear transient analysis

The approximate integral operator extended to a class of nonlinear transient thermal analysis situations described by $\mathbf{C}(\vec{T})\dot{\vec{T}} + f(\vec{T}) = \vec{F}(t)$, with the need to compute the eigenproblem only once based on the initial state is briefly highlighted next. For nonlinear situations, the approximate integral operator is

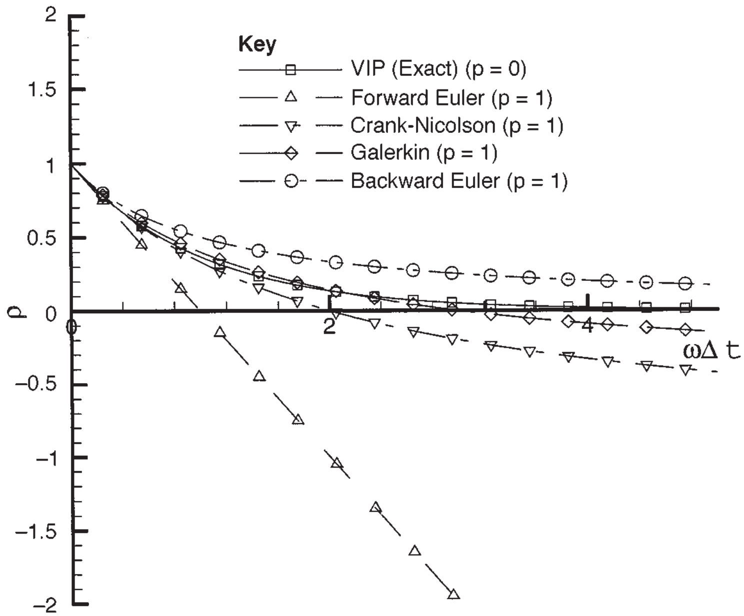

Figure 1. Comparative stability/ accuracy characteristics

however, explicit with iterations, and for linear situations, it readily reduces to the linear time integral operator.

For nonlinear transient thermal analysis, representing

$$\mathbf{C}\left(\vec{T}\right)\dot{\vec{T}} + \vec{f}\left(\vec{T}\right) = \vec{F}\left(\vec{T}\right) \qquad \vec{T}(0) = \vec{T}_0 \tag{13}$$

where $\vec{f}(\vec{T}) = [\mathbf{K}_c(\vec{T}) + \mathbf{K}_h(\vec{T}) + \mathbf{K}_r(\vec{T})]\vec{T}$ arises due to contributions from conduction, convection, and radiation, and the load $\vec{F}$ comprises of the corresponding contributions to the load vector to also include heat generation. For the purpose of the theoretical development of the time integral operator, considering (for illustration only)

$$\mathbf{C} = \mathbf{C}_L + \mathbf{C}_{NL} \tag{14}$$

and introducing the above into the semi-discretized system, we have at an arbitrary time level, γ,

$$\mathbf{C}_L\dot{\vec{T}}^{\gamma} + \vec{f}^{\gamma} = \vec{F}^{\gamma} - \mathbf{C}_{NL}^{\gamma} - \mathbf{C}_{NL}^{\gamma}\dot{\vec{T}}^{\gamma} = \vec{R}_{eq}^{\gamma} \tag{15}$$

We now invoke

$$\int_{t_n}^{t_{n+1}} \mathbf{W}^T\left[C_L\dot{\vec{T}}^{\gamma} + \vec{f}^{\gamma} - \vec{R}_{eq}^{\gamma}\right]d\bar{\tau} = 0 \tag{16}$$

and consistently integrate by parts the first time derivative term. Following analogous procedures outlined earlier and selecting the virtual or weighted time field as the solution of the following equation based on the initial state,

$$\dot{\mathbf{W}}^T\mathbf{C}_L - \mathbf{W}^T\mathbf{K}^0 = \mathbf{W}^T(\tau)\mathbf{C}_L\delta(t-\tau) \tag{17}$$

an explicit time integral methodology with iterations is obtained as described by Tamma *et al.* (1994, 1995), Chen *et al.* (1993) and Mei *et al.* (1994). For linear situations, it can be shown to reduce to either the exact or the approximate integral operator based on the approximation introduced for the load. Figure 2 illustrates a typical linear/nonlinear transient thermal analysis application via the explicit VIP time integral methodology. For given accuracy considerations, of noteworthy mention is the naturally inherent stability advantage of the time integral operator. The computational effectiveness depends upon the problem at hand and the pros and cons are outlined in Tamma *et al.* (1997) and references thereof.

In closure, the time integral or equivalently the modal based time operator described in this section has been shown to be simply an outcome of a generalized time weighted philosophy with the characterization that the

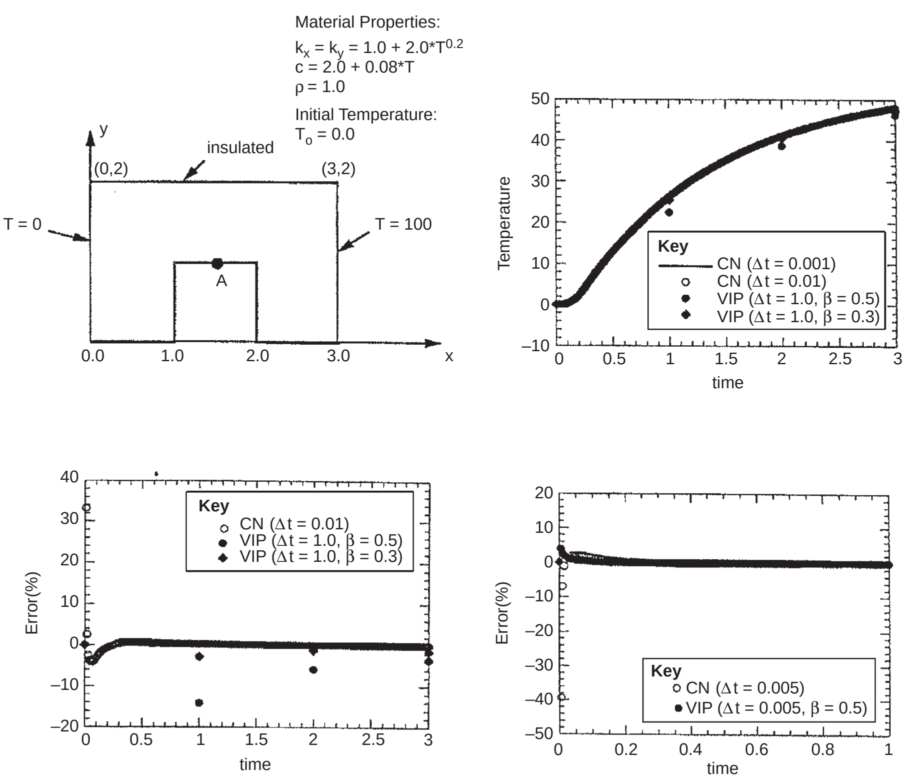

Figure 2.
Plate with hole; comparative temperature distributions and errors

weighted time fields are theoretically exact ($\mathbf{W} = \mathbf{W}_{Exact}$) and no conditions need to be imposed on the dependent field variable, that is, consideration of the temperature field approximation is irrelevant. In the next section, the formal development of integration operators in time and their characterization are presented and they emanate directly from approximations introduced in the development of integral operators in time described in this section.

Formal development of time integration operators from time integral operators

Development of single-step integration operators [SSInO]

Although an explicit time integral methodology of computation was outlined earlier employing the theoretical weighted time fields for linear and a class of nonlinear situations, some analysts may not wish to employ integral operators, [IO]'s of Type 1 classification primarily because of the need and the expense of solving the associated eigenproblem, and/or stem concerns for general transient analysis applications. As such integration operators [InO]'s, have been the selected choices. A plausible theory of evolution and the formal development of such time integration operators, of Type 3 classification, directly emanating from introducing approximations in the previous development of time integral operators via a degenerated representation of the matrix form of the theoretical weighted time fields which does not preserve the theoretical form is briefly outlined next. Elsewhere, a detailed description of Type 2 time operators are presented by Tamma *et al.* (1999).

In lieu of the existence of a variety of parameters present in the various algorithms available in the literature and to minimize notational confusion which deters lucid communication when referring to and relating the different algorithms, a standardized formal ideology is described for further characterizing a wide class of algorithms pertaining to the Type 3 classification. As such, discrete numerically assigned [DNA] algorithmic markers serve well to uniquely characterize algorithms. Additionally, a logical sequence of evolution of single-step integration operators [SSInO] and consequences leading to multi-step integration operators [MSInO], and a class of finite element in time integration operators [FETInO] and their relationships are described.

The bridging of the relationships between integral operators in time described previously to time integration operators follows next. From the theoretical weighted time fields, the degenerated representation of the weighted time field is formulated as follows. For the multi-degree of freedom transient problem, merely for the purposes of the development of the theory, and to provide a plausible explanation of how various polynomial forms of the weighted time fields have evolved from the exact weighted time fields, consider the theoretical approximation (in a mean sense) of all the system eigenvalues and designated as λ_m given by:

$$\lambda_m = \frac{1}{n}\sum_{i=1}^{n} \lambda_i \tag{18}$$

where λ_m is a theoretical quantity of the mean of all the system eigenvalues. Employing this approximation in the theoretical weighted time field expression described earlier results in

$$W(t) = e^{\lambda_m (t-\tau)} H(t-\tau) \tag{19}$$

where $\tau \in [t_n, t_{n+1}]$. Note that W(t) is now approximated instead as a single valued scalar function of time which does not preserve the original matrix representation of the theoretical weighted time fields, $\mathbf{W}_{Exact}$. It is to be noted that the result leading to equation (19) is simply a plausible theoretical explanation from which various approximations for the weighted time field can be shown to have evolved and no eigenvalues or eigen problem is really computed. The implication is that it is this degenerated scalar form of the weighted time field which can explain, for example, the reasons leading to the consideration of a pth order asymptotic type series approximation for the weighted time field in the subsequent development of time integration operators of Type 3 classification. For illustration, consider now a thought experiment by employing an asymptotic type series expansion for $W(t) = e^{\lambda_m (t-\tau)} H(t-\tau)$ resulting in:

$$W_{Asymp}(t) = \left[w_0 + w_1 \left(\frac{t-\tau}{\Delta t} \right) + w_2 \left(\frac{t-\tau}{\Delta t} \right)^2 + \ldots + w_p \left(\frac{t-\tau}{\Delta t} \right)^p \right] H(t-\tau) \tag{20}$$

Setting $\tau = t_n$, leads to:

$$W_{Asymp}(t) = w_0 + w_1 \left(\frac{t-t_n}{\Delta t} \right) + w_2 \left(\frac{t-t_n}{\Delta t} \right)^2 + \ldots + w_p \left(\frac{t-t_n}{\Delta t} \right)^p \tag{21}$$

Letting $t - t_n = \bar{\tau}$, and denoting $\Gamma = \frac{t-t_n}{\Delta t}$, yields the generalized approximated virtual or weighted time field for a wide class of integration operators:

$$W_{Asymp}(t) = w_0 + w_1 \Gamma + w_2 \Gamma^2 + \ldots + w_p \Gamma^p \tag{22}$$

At this juncture, one cannot now disregard the corresponding approximation imposed upon $\vec{T}$ as was the case with the selection of the theoretical virtual or weighted time field which did not impose any conditions on the dependent field variable since it was irrelevant. A consistent choice for $\vec{T}$ needs to be made as:

$$\vec{T}_{Asymp} = \vec{T}_n + \beta_1 \dot{\vec{T}}_n \bar{\tau} + \beta_2 \ddot{\vec{T}}_n \bar{\tau}^2 + \ldots + \beta_p \vec{T}^{(p)} \bar{\tau}^p \tag{23}$$

for which (as a particular case) the Taylor series is given as

$$\vec{T}_{Taylor} = \vec{T}_n + \dot{\vec{T}}_n \bar{\tau} + \ddot{\vec{T}}_n \frac{\bar{\tau}^2}{2!} + \ldots + \vec{T}^{(p)} \frac{\bar{\tau}^p}{p!} \tag{24}$$

Substituting the above approximations for $W_{Asymp}(t)$ and $\vec{T}_{Taylor}$ into the time weighted residual semi-discretized transient field problem, yields a generalized family of single-step integration operators [SSInO]:

$$\left[\sum_{q=1}^{p}\frac{\Delta t^{q-1}\vec{T}_n^{(q)}}{(q-1)!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i+1}\right]\mathbf{C}+\left[\sum_{q=0}^{p}\frac{\Delta t^q\vec{T}_n^{(q)}}{q!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i}\right]\mathbf{K} - \sum_{q=0}^{p}\int_0^{\Delta t} w_q\bar{\tau}^q\vec{F}_n d\bar{\tau}=0 \tag{25}$$

As such, it is herein demonstrated that the above SSInO family of algorithms are an outcome of a weighted time philosophy employing $W(t) = W_{Asymp}$ (degenerated from $\mathbf{W}_{Exact}$) with the imposed conditions for the dependent field variable as $\vec{T} = \vec{T}_{Taylor}$ (issues concerning $\vec{T}_{Asymp}$ are not discussed here). Equation (25) allows the computation of $\vec{T}^{(p)}$when all the lower order derivatives of the temperature and the distinct values of the coefficients of w_i are known. Once $\vec{T}^{(p)}$ is found, the temperature vector at $t = t_{n+1}$ is determined from:

$$\vec{T}_{n+1} = \vec{T}_n + \dot{\vec{T}}_n\Delta t + \frac{1}{2!}\ddot{\vec{T}}_n\Delta t^2 + \ldots + \frac{1}{p!}\vec{T}^{(p)}\Delta t^p \tag{26}$$

In the above, there exist p unknowns and one equation, where $\dot{\vec{T}}_0, \ddot{\vec{T}}_0, \cdots, \vec{T}_0^{(p)}$ are the unknowns. $\vec{T}_0$ can be obtained from the semi-discretized system based on the initial conditions. The p-th derivative $\vec{T}^{(p)}$ can be obtained from equation (25). For $p > 2$, $\ddot{\vec{T}}_0, \dddot{\vec{T}}, \ldots, \vec{T}_0^{(p-1)}$ need to be calculated. Based on accuracy and/or efficiency considerations, one may utilize the differential equation system or alternatives as described in Zienkiewicz and Taylor (1994) and Tamma *et al.* (1997).

A generalized implementation procedure is outlined next for the W_p-family of integral operators [IO] with $p = 0$, and the single-step integration operators [SSInO] with $p = 1, 2, 3, \ldots,$.

Generalized implementation: W_p algorithms for transient analysis

Step 1. Evaluate the finite element **C** and **K** matrices.

Step 2. Select option for **W**: If exact or approximation integral operator, evaluate:

$$\mathbf{W}_{Exact} = diag\left[e^{\lambda_1\bar{\tau}}, e^{\lambda_2\bar{\tau}}, \ldots, e^{\lambda_n\bar{\tau}}\right]$$

Solve eigenproblem based on conditions at initial state $t = t_n$:

$$\mathbf{KX} = \mathbf{CX}\Omega$$

Determine initial conditions in modal coordinates:

$$\vec{\Theta}(0) = \mathbf{X}^T\mathbf{C}\vec{T}(0)$$

go to step 6, else select coefficients w_i:

$$W_{Asymp} = w_0 + w_1\Gamma + w_2\Gamma^2 + \ldots + w_p\Gamma^p$$

Step 3. Assume a temperature field described by:

$$\vec{T}_{Taylor} = \vec{T}_n + \dot{\vec{T}}_n\bar{\tau} + \cdots + \frac{1}{p!}\vec{T}^{(p)}\bar{\tau}^p$$

Step 4. If $p > 1$, compute $\dot{\vec{T}}_0$ from initial conditions.
Step 5. If $p > 2$, solve for $\ddot{\vec{T}}_0, \vec{T}_0, \cdots, \vec{T}_0{}^{(p-1)}$. Based on accuracy and/or efficiency considerations, perform evaluations as discussed previously.
Step 6. Enter time step loop. If exact or approximate integral operator: Compute $\vec{F}^* = X^T\vec{F}$, form integral operator and solve explicitly:

$$\vec{\Theta}_{n+1,i} = A_{amp}\vec{\Theta}_{n,i} + \vec{L}$$

goto step 9, else form the time operator, and solve for $\vec{T}^{(p)}$:

$$\left[\sum_{q=1}^{p}\frac{\Delta t^{q-1}\vec{T}_n^{(q)}}{(q-1)!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i+1}\right]\mathbf{C} + \left[\sum_{q=0}^{p}\frac{\Delta t^q\vec{T}_n^{(q)}}{q!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i}\right]\mathbf{K} - \sum_{q=0}^{p}\int_0^{\Delta t} w_q\bar{\tau}^q\vec{F}_n d\bar{\tau} = 0$$

Step 7. Obtain:

$$\vec{T}_{n+1} = \vec{T}_n + \dot{\vec{T}}_n\Delta t + \cdots\frac{\vec{T}^{(p)}\Delta t^p}{p!}$$

Step 8. Update:

$$\dot{\vec{T}}_{n+1} = \dot{\vec{T}}_n + \Delta t\ddot{\vec{T}}_n + \cdots + \frac{\Delta t^{p-1}}{(p-1)!}\vec{T}^{(p)}$$

$$\ddot{\vec{T}}_{n+1} = \ddot{\vec{T}}_n + \Delta t\,\dddot{\vec{T}}_n + \cdots + \frac{\Delta t^{p-2}}{(p-1)!}\vec{T}^{(p)}$$

$$\vdots$$

$$\vec{T}_{n+1}^{(p-1)} = \vec{T}_n{}^{(p-1)} + \Delta t\vec{T}^{(p)}$$

goto step 6 for the next time step.

Step 9. Obtain:

$$\vec{T}_{n+1} = \mathbf{X}\vec{\Theta}_{n+1}$$

goto step 6 for the next time step.

Tables I-IV specifically identify the weighted time fields for a majority of the W_p family of algorithms for first-order systems. For $p = 1, 2, 3$ and 4 and the like, one can readily characterize transient algorithms via discrete numerically assigned [DNA] markers for the coefficients of w_i and the corresponding approximation for the dependent field variable, thus leading to a variety of algorithms as choices available to the analyst. The development of some of the

Table I. Type 3-[DNA] algorithmic markers for W_1 algorithms with $\vec{T}_{Linear}$

Algorithms	Weighted time fields
Crank-Nicolson (1947)	$W = 1 + 0\Gamma = 1$
Euler Forward	$W = 1 - \frac{3}{2}\Gamma$
Euler Backward	$W = 1 - 3\Gamma$
Liniger (1968)	$W = 1 - \frac{17}{4}\Gamma$
Galerkin (Zlamal, 1977)	$W = 1 - \infty\Gamma$

Table II. Type 3-[DNA] algorithmic markers for W_2 algorithms with $\vec{T}_{Quadratic}$

Algorithms	Weighted time fields
Dupont *et al.* (1974)	$W = 1 - \frac{19}{3}\Gamma + \frac{20}{3}\Gamma^2$
Gear (1969)	$W = 1 - \frac{36}{5}\Gamma + 8\Gamma^2$
Lees (1966)	$W = 1 - \frac{60}{11}\Gamma + \frac{60}{11}\Gamma^2$
Liniger-1 (1969)	$W = 1 + 146.182\Gamma - 279.091\Gamma^2$
Liniger-2 (1969)	$W = 1 - 8.88424\Gamma + 11.0961\Gamma^2$
Zlamal (1977)	$W = 1 - \frac{236}{39}\Gamma + \frac{80}{13}\Gamma^2$

Table III. Type 3-[DNA] algorithmic markers for W_3 algorithms with $\vec{T}_{Cubic}$

Algorithms	Weighted time fields
Gear (1969)	$W = 1 - \frac{615}{46}\Gamma + \frac{825}{23}\Gamma^2 - \frac{1155}{46}\Gamma^3$
Liniger (1969)	$W = 1 - 13.8673\Gamma + 38.1061\Gamma^2 - 27.1143\Gamma^3$
Zlamal (1977)	$W = 1 - 16.8806\Gamma + 49.5224\Gamma^2 - 36.5672\Gamma^3$

Table IV. Type 3-[DNA] algorithmic markers for W_4 algorithms with $\vec{T}_{Quartic}$

Algorithms	Weighted time fields
Gear (1969)	$W = 1 - \frac{6495}{302}\Gamma + \frac{30765}{302}\Gamma^2 - \frac{24850}{151}\Gamma^3 + \frac{12789}{151}\Gamma^4$

[DNA]-algorithmic markers have been facilitated from results reported in Zienkiewicz and Taylor (1994). These are summarized in Tables I to IV, where $\Gamma = \frac{\bar{\tau}}{\Delta t}$.

Development of multi-step integration operators [MSInO]

In the previous section, a W_p-family of Type 3 classification of single-step time integration operators [SSInO] have been constructed for $p = 1, 2, 3, \cdots$, employing $W_{Asymp}(t)$ and $\vec{T}_{Taylor}$ directly from the developments described for Type 1 classification of time integral operators ($p = 0$) which employed the theoretical weighted time fields, $\mathbf{W}_{Exact}$ and with no specific conditions imposed for $\vec{T}$ as it is irrelevant. In the previous single-step algorithms, the unknown value, $\vec{T}_{n+1}$, of the time step t_{n+1} is related to the known values, $\vec{T}_n, \dot{\vec{T}}_n, \cdots, \vec{T}_n^{(p)}$, of the time step t_n. Instead of a single time interval t_n to t_{n+1}, the same philosophy and consequences drawn in relation to multi-step representations is described next now employing W_{Asymp} and $\vec{T}_{Lagrange}$ constructed over the multi-steps $n - p + 1$ to $n + 1$. Although in principle, analogous related efforts are described in Zienkiewicz and Taylor (1994), the present developments provide the underlying theoretical basis and further insight into additionally explaining the multi-step time integration operators within the context of the proposed overall framework. In general, the higher-order derivatives at time step t_n, namely, $\dot{\vec{T}}_n, \ddot{\vec{T}}_n, \cdots, \vec{T}_n^{(p)}$ in SSInO can be expressed in relation to Lagrange polynomial approximations in terms of $\vec{T}_{n-p+1}, \vec{T}_{n-p+2}, \cdots, \vec{T}_{n+1}$. When these relations are substituted into the temperature field approximation associated with the single-step representations, and the dependent field variable is now constructed over p-steps as $\vec{T}_{Lagrange}$, then in conjunction with the degenerated weighted time field approximation, W_{Asymp}, the resulting representations will now yield the corresponding multi-step forms. Thereby, showing that the single-step integration operators [SSInO] described previously can be readily cast as and are equivalent to the multi-step counterparts [MSInO] as presented next. Of interest, however, are also the subtle issues as related to the accuracy of these multi-step results in contrast to the single-step counterparts, and conditions under which equivalence can be clearly explained.

W_p Family	Type	Stability	Accuracy
VIP (p=0)	Explicit	Unconditional	2
Forward Euler (p=1)	Explicit	Conditional	1
Crank-Nicolson (p=1)	Implicit	Unconditional	2
Galerkin (p=1)	Implicit	Unconditional	1
Backward Euler (p=1)	Implicit	Unconditional	1

Table V. Comparative stability/accuracy characteristics

General multi-step algorithms

Summarizing the procedures of the W_p family of single-step integration operators, for the semi-discretized first order equation:

$$\mathbf{C}\dot{\vec{T}} + \mathbf{K}\vec{T} = \vec{F} \tag{27}$$

with initial condition

$$\vec{T}(0) = \vec{T}_0 \tag{28}$$

we had derived a weighted time field (as an approximation resulting from the theoretical weighted time field) of the form:

$$W_{Asymp}(\bar{\tau}) = w_0 + w_1\Gamma + \cdots + w_p\Gamma^p \tag{29}$$

with the imposed conditions for the selection of $\vec{T}$ as a function between t_n and t_{n+1} as:

$$\vec{T}_{Taylor} = \vec{T}_n + \dot{\vec{T}}_n\bar{\tau} + \ddot{\vec{T}}_n\frac{\bar{\tau}^2}{2!} + \cdots + \vec{T}_n^{(p)}\frac{\bar{\tau}^p}{p!} \tag{30}$$

with $\bar{\tau} \in [0, \Delta t]$. Substituting W_{Asymp} and $\vec{T}_{Taylor}$ into the time weighted residual semi-discretized transient problem yields the family of single-step integration operators [SSInO] as:

$$\left[\sum_{q=1}^{p}\frac{\Delta t^{q-1}\vec{T}_n^{(q)}}{(q-1)!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i+1}\right]\mathbf{C} + \left[\sum_{q=0}^{p}\frac{\Delta t^q\vec{T}_n^{(q)}}{q!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i}\right]\mathbf{K} - \sum_{q=0}^{p}\int_0^{\Delta t} w_q\bar{\tau}^q\vec{F}_n d\bar{\tau} = 0 \tag{31}$$

As described in the previous section, equation (31) represents the general single-step family of time integration operators [SSInO]. Instead, following the previous discussion, now consider constructing $\vec{T}$ over p-steps as a polynomial in terms of $\vec{T}_{n-p+1}, \vec{T}_{n-p+2}, \cdots, \vec{T}_{n+1}$. Thus,

$$\vec{T} = F\left(\vec{T}_{n+1}, \vec{T}_n, \cdots, \vec{T}_{n-p+1}, \bar{\tau}\right) \tag{32}$$

In order to construct the above, we consider the approximation of the derivatives $\dot{T}_n, \ddot{T}_n, \cdots, T_n^{(p)}$ via Lagrange interpolation functions following Zienkiewicz and Taylor (1994), but then introduce these into the representation for $\vec{T}_{Taylor}$ given by equation (30). Hence we have upon employing

$$\begin{aligned} \dot{\vec{T}}_n &\approx \sum_{i=-p+1}^{1} \dot{N}_i(0)\vec{T}_{n+i} \\ \ddot{\vec{T}}_n &\approx \sum_{i=-p+1}^{1} \ddot{N}_i(0)\vec{T}_{n+i} \\ &\vdots \\ \vec{T}_n^{(p)} &\approx \sum_{i=-p+1}^{1} N_i^{(p)}(0)\vec{T}_{n+i} \end{aligned} \tag{33}$$

where, in general at $\bar{\tau}$ we have

$$N_i^{(p)}(\bar{\tau}) = \sum_{\substack{k=-p+1 \\ k\neq i}}^{1} \cdots \sum_{\substack{l=-p+1 \\ l\neq k \\ \vdots \\ l\neq i}}^{1} \frac{\prod\limits_{\substack{m=-p+1 \\ m\neq k \\ m\neq i \\ \vdots \\ m\neq l}}^{1} (\bar{\tau} - m\Delta t)}{\prod\limits_{\substack{k=-p+1 \\ k\neq i}}^{1} (i-k)\Delta t}$$

The temperature field $\vec{T}_{Taylor}$ is now approximated as

$$\begin{aligned} \vec{T}_{Taylor} \approx& \vec{T}_n + \bar{\tau} \sum_{i=-p+1}^{1} \dot{N}_i(0)\vec{T}_{n+i} + \frac{1}{2!}\bar{\tau}^2 \sum_{n=-p+1}^{1} \ddot{N}_i(0)\vec{T}_{n+i} \\ &+ \cdots + \frac{1}{p!}\bar{\tau}^p \sum_{i=-p+1}^{1} N_i^{(p)}(0)\vec{T}_{n+i} \end{aligned} \tag{34}$$

Re-arranging equation (34), yields:

$$\begin{aligned} \vec{T}_{Taylor} &\approx \sum_{i=-p+1}^{1} N_i(\bar{\tau})\vec{T}_{n+i} \\ &= \sum_{i=-p+1}^{1} \prod_{\substack{k=-p+1 \\ k\neq i}}^{1} \frac{\bar{\tau} - k\Delta t}{(i-k)\Delta t} \vec{T}_{n+i} \\ &= F\left(\vec{T}_{n+1}, \vec{T}_n, \cdots, \vec{T}_{n-p+1}, \bar{\tau}\right) \end{aligned} \tag{35}$$

Finally,

$$\vec{T}_{Taylor} = \vec{T}_n + \bar{\tau}\dot{\vec{T}}_n + \frac{1}{2!}\bar{\tau}^2\ddot{\vec{T}}_n + \cdots + \frac{1}{p!}\bar{\tau}^p\vec{T}_n^{(p)}$$
$$\approx \sum_{i=-p+1}^{1} \prod_{\substack{k=-p+1 \\ k\neq i}}^{1} \frac{\bar{\tau} - k\Delta t}{(i-k)\Delta t}\vec{T}_{n+i} = \vec{T}_{Lagrange} \tag{36}$$

implying that both $\vec{T}_{Taylor}$ and $\vec{T}_{Lagrange}$ are approximated to p^{th} order; however $\vec{T}_{Taylor} \neq \vec{T}_{Lagrange}$. In a similar manner, we have the higher order derivative approximations as:

$$\dot{\vec{T}}_{Taylor} = \dot{\vec{T}}_n + \bar{\tau}\ddot{\vec{T}}_n + \cdots + \frac{1}{(p-1)!}\bar{\tau}^{p-1}\vec{T}_n^{(p)}$$
$$\approx \sum_{i=-p+1}^{1}\dot{N}_i(0)\vec{T}_{n+i} + \bar{\tau}\sum_{i=-p+1}^{1}\ddot{N}_i(0)\vec{T}_{n+i}$$
$$+ \cdots + \frac{1}{(p-1)!}\bar{\tau}^{p-1}\sum_{i=-p+1}^{1}N_i^{(p)}(0)\vec{T}_{n+i} \tag{37}$$
$$= \sum_{i=-p+1}^{1}\dot{N}_i(\bar{\tau})\vec{T}_{n+i} = \dot{\vec{T}}_{Lagrange}$$

with the other derivatives given by

$$\ddot{\vec{T}}_{Taylor} = \ddot{\vec{T}}_n + \bar{\tau}\,\dddot{\vec{T}}_n + \cdots + \frac{1}{(p-2)!}\bar{\tau}^{p-2}\vec{T}_n^{(p)}$$
$$\approx \sum_{i=-p+1}^{1}\ddot{N}_i(0)\vec{T}_{n+i} + \bar{\tau}\sum_{i=-p+1}^{1}\dddot{N}_i(0)\vec{T}_{n+i} + \cdots + \frac{1}{(p-2)!}\bar{\tau}^{p-2}$$
$$\sum_{i=-p+1}^{1}N_i^{(p)}(0)\vec{T}_{n+i}$$
$$= \sum_{i=-p+1}^{1}\ddot{N}_i(\bar{\tau})\vec{T}_{n+i}$$
$$\vdots$$

$$\vec{T}^{(p-1)}_{Taylor} = \vec{T}^{(p-1)}_{n} + \bar{\tau}\vec{T}^{(p)} \approx \sum_{i=-p+1}^{1} N_i^{(p-1)}(0)\vec{T}_{n+i} + \bar{\tau}\sum_{i=-p+1}^{1} N_i^{(p)}(0)\vec{T}_{n+i} = \sum_{i=-p+1}^{1} N_i^{(p-1)}(\bar{\tau})\vec{T}_{n+i} \quad (38)$$

Employing the Lagrange approximations associated with the dependent field variable in equation (36) and (37) along with the same degenerated weighted time field, W_{Asymp}, into the time weighted residual semi-discretized transient problem, we now have the resulting multi-step integration operators [MSInO] as:

$$\int_0^{\Delta t} W(\bar{\tau}) \left[\mathbf{C}\sum_{i=-p+1}^{1} \dot{N}_i(\bar{\tau})\vec{T}_{n+i} + \mathbf{K}\sum_{i=-p+1}^{1} N_i(\bar{\tau})\vec{T}_{n+i} - \sum_{i=-p+1}^{1} N_i(\bar{\tau})f_{n+i}\right] d\bar{\tau} = 0 \quad (39)$$

or

$$\left[\sum_{j=0}^{p-1}\sum_{i=-p+1}^{1}\int_0^{\Delta t} w_j\Delta t^j \dot{N}_i(\bar{\tau})\vec{T}_{n+i}d\bar{\tau}\right]\mathbf{C} + \left[\sum_{j=0}^{p-1}\sum_{i=-p+1}^{1}\int_0^{\Delta t} w_j\Delta t^j N_i(\bar{\tau})\vec{T}_{n+i}d\bar{\tau}\right]\mathbf{K} - \sum_{j=0}^{p-1}\sum_{i=-p+1}^{1}\int_0^{\Delta t} w_j\Delta t^j N_i(\bar{\tau})f_{n+i}d\bar{\tau} = 0 \quad (40)$$

or

$$\left[\sum_{q=1}^{p}\frac{\Delta t^{q-1}\sum_{j=-p+1}^{1} N_j^q\vec{T}_{n+j}}{(q-1)!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i+1}\right]\mathbf{C} + \left[\sum_{q=0}^{p}\frac{\Delta t^{q-1}\sum_{j=-p+1}^{1} N_j^q\vec{T}_{n+j}}{q!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i}\right]\mathbf{K} - \sum_{q=0}^{p}\int_0^{\Delta t} w_q\bar{\tau}^q\vec{F}_n d\bar{\tau} = 0 \quad (41)$$

Next the underlying subtle issues contrasting SSInO and the MSInO and the conditions under which equivalence can be demonstrated are explained. The resulting multi-step representations, namely, [MSInO], in principle can be solved in two ways. The first approach follows the implementation steps of [SSInO], and solving the corresponding higher-order derivative quantity $\sum_{i=-p+1}^{1} N_i^{(p)}(0)\vec{T}_{n+i}\left(\approx \vec{T}_n^{(p)}\right)$ employing equation (41). It should be noted that in the first approach $\vec{T}_{n+1}$ is not directly solved from equation (41). The corresponding implementation steps are as follows:

Step 1. Evaluate the finite element $\mathbf{C}$ and $\mathbf{K}$ matrices.
Step 2. Select coefficients w_i 's for the weighted time field, W_{Asymp}:

$$W_{Asymp} = w_0 + w_i\Gamma + w_2\Gamma^2 + \cdots + w_p\Gamma^p$$

Step 3. Assume the temperature field described by:

$$\vec{T}_{Lagrange} = \sum_{i=-p+1}^{1} N_i(\bar{\tau})\vec{T}_{n+i}$$

where N_i are the Lagrange interpolation functions:

$$N_i = \prod_{\substack{k=-p+1\\ k\neq i}}^{1} \frac{\bar{\tau} - k\Delta t}{(i-k)\Delta t}$$

Step 4. Compute $\dot{\vec{T}}_0, \ddot{\vec{T}}_0, \cdots, \vec{T}_0^{(p)}$ from initial conditions by using:

$$\mathbf{C}\vec{T}_0^i = \vec{F}_0^{i-1} - \mathbf{K}\vec{T}_0^{i-1}$$

Step 5. Solve initial back steps $\vec{T}_{-p+1}, \vec{T}_{-p+2}, \cdots, \vec{T}_{-1}$, from:

$$\dot{\vec{T}}_0 = \sum_{i=-p+1}^{1} \dot{N}_i(0)\vec{T}_i$$

$$\ddot{\vec{T}}_0 = \sum_{i=-p+1}^{1} \ddot{N}_i(0)\vec{T}_i$$

$$\vdots$$

$$\vec{T}_0^{(p)} = \sum_{i=-p+1}^{1} N_i^{(p)}(0)\vec{T}_i$$

Step 6. Enter time step loop. Solve for the quantity $\sum_{i=-p+1}^{1} N^{(p)}(0)\ \vec{T}_{n+i}$ from:

$$\left[\sum_{q=1}^{p}\frac{\Delta t^{q-1}\sum_{j=-p+1}^{1}N_j^{(q)}\vec{T}_{n+j}}{(q-1)!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i+1}\right]\mathbf{C}$$
$$+\left[\sum_{q=0}^{p}\frac{\Delta t^{q-1}\sum_{j=-p+1}^{1}N_j^{(q)}\vec{T}_{n+j}}{q!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i}\right]\mathbf{K}$$
$$-\sum_{q=0}^{p}\int_0^{\Delta t}w_q\bar{\tau}^q\vec{F}_n d\bar{\tau}=0$$

Step 7. Obtain:

$$\vec{T}_{n+1}=\sum_{i=-p+1}^{1}N_i(0)\vec{T}_{n+i}+\Delta t\sum_{i=-p+1}^{1}\dot{N}_i(0)\vec{T}_{n+i}+\cdots+\frac{\Delta t^p}{p!}\sum_{i=-p+1}^{1}N_i^{(p)}(0)\vec{T}_{n+i}$$

Step 8. Update:

$$\sum_{i=-p+1}^{1}\dot{N}_i(0)\vec{T}_{n+1+i}=\sum_{i=-p+1}^{1}\dot{N}_i(0)\vec{T}_{n+i}+\Delta t\sum_{i=-p+1}^{1}\ddot{N}_i(0)\vec{T}_{n+i}+\cdots+\frac{\Delta t^p}{p!}\sum_{i=-P+1}^{1}N_i^{(p)}(0)\vec{T}_{n+i}$$
$$\vdots$$
$$\sum_{i=-p+1}^{1}N_i^{(p-1)}(0)\vec{T}_{n+1+i}=\sum_{i=-p+1}^{1}N_i^{(p-1)}(0)\vec{T}_{n+i}+\Delta t\sum_{i=-p+1}^{1}N_i^{(p)}(0)\vec{T}_{n+i}$$

Step 9. Re-do the time step loop for next time step.

In [SSInO], the temperature field was approximated as a pth order approximation of the Taylor series expansion, while it is approximated as Lagrange interpolation functions also up to p^{th} order in [MSInO]. In the previous [MSInO] implementation procedure, the higher order derivatives, $\dot{\vec{T}}_n, \ddot{\vec{T}}_n, \cdots, \vec{T}_n^{(p)}$ correspond to $\sum_{i=-p+1}^{1}\dot{N}_i(0)\vec{T}_{n+i}, \sum_{i=-p+1}^{1}\ddot{N}_i(0)\vec{T}_{n+i}, \cdots, \sum_{i=-p+1}^{1}N_i^{(p)}(0)\vec{T}_{n+i}$, as shown in equation (33). By choosing the same weighted time fields for [SSInO] and MSInO], namely, W_{Asymp}, this implementation procedure forces the two assumed temperature fields to be the same, which means that the implementation procedure enforces $\sum_{i=-p+1}^{1}N_i^{(j)}\vec{T}_{n+i}\equiv\vec{T}_n^{(j)}$, where $j=1,2,\cdots,p$, and hence the resulting solutions in [SSInO] and [MSInO] are identical. Hence, with the same weighted time field, the resulting multi-step representations, equation (41), inherit the same stability, convergence properties and numerical solution as the corresponding single-step representations, equation (31), since essentially there is no difference in that the same quantities are being identically computed (note that the initial conditions are also set to be the same).

Alternatively, in the multi-step integration operators, equation (41), there exists only the unknown term $\vec{T}_{n+1}$. In this case, $\vec{T}_{n+1}$ can be also directly solved from equation (41), instead of solving for the quantity $\sum_{i=-p+1}^{1} N_i^{(p)} \vec{T}_{n+i}$ as in the previous approach. The alternative implementation procedural steps are described next:

Step 1. Evaluate the finite element **C** and **K** matrices.
Step 2. Select coefficients w_i's for weighted time field, W_{Asymp}:

$$W_{Asymp} = w_0 + w_i\Gamma + w_2\Gamma^2 + \cdots + w_p\Gamma^p$$

Step 3. Assume the temperature field described by:

$$\vec{T}_{Lagrange} = \sum_{i=-p+1}^{1} N_i(\bar{\tau})\vec{T}_{n+i}$$

where N_i are Lagrange interpolation functions:

$$N_i = \prod_{\substack{k=-p+1 \\ k\neq i}}^{1} \frac{\bar{\tau} - k\Delta t}{(i-k)\Delta t}$$

Step 4. Compute $\dot{\vec{T}}_0, \ddot{\vec{T}}_0, \cdots, \vec{T}_0^{(p)}$ from initial conditions by using:

$$\mathbf{C}\vec{T}_0^{(i)} = \vec{F}_0^{(i-1)} - \mathbf{K}\vec{T}_0^{(i-1)}$$

Step 5. Solve initial back steps $\vec{T}_{-p+1}, \vec{T}_{-p+2}, \cdots, \vec{T}_{-1}$, from:

$$\dot{\vec{T}}_0 = \sum_{i=-p+1}^{1} \dot{N}_i(0)\vec{T}_i$$

$$\ddot{\vec{T}}_0 = \sum_{i=-p+1}^{1} \ddot{N}_i(0)\vec{T}_i$$

$$\cdots$$

$$\vec{T}_0^{(p)} = \sum_{i=-p+1}^{1} N_i^{(p)}(0)\vec{T}_i$$

Step 6. Enter time step loop. Solve for the term $\vec{T}_{n+1}$ from:

$$\left[\sum_{q=1}^{p}\frac{\Delta t^{q-1}\sum_{j=-p+1}^{1}N_j^{(q)}\vec{T}_{n+j}}{(q-1)!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i+1}\right]\mathbf{C}$$
$$+\left[\sum_{q=0}^{p}\frac{\Delta t^{q-1}\sum_{j=-p+1}^{1}N_j^{(q)}\vec{T}_{n+j}}{q!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i}\right]\mathbf{K}$$
$$-\sum_{q=0}^{p}\int_0^{\Delta t}w_q\bar{\tau}^q\vec{F}_n d\bar{\tau}=0$$

Step 7. Re-do the time step loop for next time step.

As described previously, the assumed temperature field, which is a function of the Lagrange interpolation functions over $\vec{T}_{-p+1},\vec{T}_{-p+2},\cdots,\vec{T}_1$, namely, $\vec{T}_{Lagrange}$ is different from the assumed temperature field $\vec{T}_{Taylor}$ of the Taylor series expansion in [SSInO]. This causes the computed quantity, $\vec{T}_{n+1}$ to be different in both if the same weighted time field, W_{Asymp} is employed. An in-depth explanation of this and other issues and conditions under which they are equivalent follows. In the above procedure, the temperature field is assumed as p-th order, such that the higher-order derivative terms, namely, $\sum_{i=-p+1}^{1}\dot{N}_i\vec{T}_{n+i},\sum_{i=-p+1}^{1}\ddot{N}_i\vec{T}_{n+i},\cdots,\sum_{i=-p+1}^{1}N_i^{(p)}\vec{T}_{n+i}$, are only accurate up to p-th order. In [SSInO], the higher-order derivative terms, $\dot{\vec{T}}_n,\ddot{\vec{T}}_n,\cdots,\vec{T}_n^{(p)}$ are computed from the system of equations analogous to (26), hence they are considered as "exact" higher-order derivatives. These differences will cause the family of [MSInO] algorithms as different algorithms from the corresponding family of [SSInO] algorithms by choosing the same weighting field, W_{Asymp}. As such, the following is noteworthy.

Wood (1990) and Zienkiewicz and Taylor (1994) discuss adjusting the weighted time fields to obtain the equivalent algorithms for the single-step and multi-step representations. This is explained here by defining:

$$W_{p,j}=\int_0^{\Delta t}W\bar{\tau}^j d\bar{\tau}=\sum_{i=0}^{p}\frac{w_i\Delta t^{j+1}}{1+i+j}\tag{42}$$

such that $W_j\doteq\frac{W_{p,j}}{\Delta t^j W_{p,0}}$ (is now equivalent to the parameters Θ_j which are defined by Zienkiewicz and Taylor, 1994) and $j=1,2,\cdots,p$. Since the time integrators pertaining to SSInO yield different numerical results in contrast to that obtained by MSInO (as $\vec{T}_{Taylor}\neq\vec{T}_{Lagrange}$) for the same weighted time field, W_{Asymp}, adjusting the weighted time fields in the development of MSInO following the relations shown next will now yield the same numerical results for the same $\vec{T}_{Lagrange}$.

$$W_{j(SSInO)} = \frac{j! \int_0^{\Delta t} N_1^{(p-j)}(\frac{\bar{\tau}}{\Delta t}) W_{Adjust} d\bar{\tau}}{\int_0^{\Delta t} W_{Adjust} d\bar{\tau}} \tag{43}$$

This resulting formulation will thus permit equivalent single-step representations. In summary, employing W_{Asymp} and $\vec{T}_{Lagrange}$ leading to [MSInO] in contrast to employing W_{Asymp} and $\vec{T}_{Taylor}$ leading to [SSInO], obviously causes the two solutions to be different since $\vec{T}_{Lagrange} \neq \vec{T}_{Taylor}$. Alternatively, adjusting the weighted time field to W_{Adjust} and employing the same $\vec{T}_{Lagrange}$ now leads to the same numerical solutions for the time integrators [MSInO] and [SSInO].

Finite element in time integration operators [FETInO]

The original postulation for a class of finite element in time methods is the application of the finite element philosophy for the time discretization of the semi-discretized parabolic equation (although finite element methods were originally developed for the space discretization). The semi-discretized equation of interest here is

$$\mathbf{C}\dot{\vec{T}} + \mathbf{K}\vec{T} = \vec{F} \tag{44}$$

with initial condition

$$\vec{T}(0) = \vec{T}_0 \tag{45}$$

In contrast to the previous developments which employed W_{Asymp} and $\vec{T}_{Lagrange}$ leading to [MSInO], we next describe the formal development of a class of finite element in time operators [FETInO] employing instead the weighted time field as a vector containing Lagrange interpolation functions, $\vec{W}_{Lagrange}$, and the same dependent field approximation, $\vec{T}_{Lagrange}$. Subsequently, simply to demonstrate equivalence to the multi-step representations, we relax the conditions associated with $\vec{W}_{Lagrange}$ in the weak form.

We discretize the time domain with uniform Δt increment. For the element containing $\vec{T}_{n-p+1}, \vec{T}_{n-p+2}, \cdots, \vec{T}_{n+1}$, the polynomial approximation for $\vec{T}$ can be constructed by the Lagrange interpolation functions as:

$$\vec{T} = \begin{bmatrix} N_{-p+1} & N_{-p+2} & \cdots & N_1 \end{bmatrix} \begin{pmatrix} \vec{T}_{n-p+1} \\ \vec{T}_{n-p+2} \\ \vdots \\ \vec{T}_{n+1} \end{pmatrix} \tag{46}$$

where

$$N_i = \prod_{\substack{k=-p+1 \\ k \neq i}}^{1} \frac{\bar{\tau} - k\Delta t}{(i-k)\Delta t} \tag{47}$$

Next choosing the weighted time field as a vector containing the same interpolation functions, we have

$$\vec{W}^T_{Lagrange,BG} = \begin{bmatrix} N_{-p+1} & N_{-p+2} & \cdots & N_1 \end{bmatrix} \tag{48}$$

Substitute (46) and (48) into the time weighted residual of the semi-discretized equation leading to:

$$\int_0^{\Delta t} \vec{W}^T_{Lagrange,BG} \left[\mathbf{C}\dot{\vec{T}} + \mathbf{K}\vec{T} - \vec{F} \right] d\bar{\tau} = 0 \tag{49}$$

where

$$\dot{\vec{T}} = \begin{bmatrix} \dot{N}_{-p+1} & \dot{N}_{-p+2} & \cdots & \dot{N}_1 \end{bmatrix} \begin{pmatrix} \vec{T}_{n-p+1} \\ \vec{T}_{n-p+2} \\ \vdots \\ \vec{T}_{n+1} \end{pmatrix} \tag{50}$$

Equation (49) can be termed as and lead to the classical finite element in time Bubnov-Galerkin type formulation. Unfortunately, such a representation does not provide a general methodology for development of solution algorithms. As such, consider a generalized Petrov-Galerkin representation for the weighted time field as:

$$\vec{W}^T_{Lagrange,PG} = \begin{bmatrix} N_{-p+1}a_{-p+1} & N_{-p+2}a_{-p+2} & \cdots & N_1 a_1 \end{bmatrix} \tag{51}$$

where a_i are the scalar values for the free parameters. Equation (49) can be now represented as

$$\int_0^{\Delta t} \vec{W}^T_{Lagrange,PG} \left[\mathbf{C}\dot{\vec{T}} + \mathbf{K}\vec{T} - \vec{F} \right] d\bar{\tau} = 0 \tag{52}$$

Expanding (52) yields

$$\int_0^{\Delta t} N_{-p+1}a_{-p+1}\Big[\mathbf{C}\dot{\vec{T}}+\mathbf{K}\vec{T}-\vec{F}\Big]d\bar{\tau}=0$$
$$\int_0^{\Delta t} N_{-p+2}a_{-p+2}\Big[\mathbf{C}\dot{\vec{T}}+\mathbf{K}\vec{T}-\vec{F}\Big]d\bar{\tau}=0 \tag{53}$$
$$\vdots$$
$$\int_0^{\Delta t} N_1 a_1\Big[\mathbf{C}\dot{\vec{T}}+\mathbf{K}\vec{T}-\vec{F}\Big]d\bar{\tau}=0$$

or, in matrix form we have the representation for the generalized form of a class of finite element in time operators given by

$$\begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1p} \\ B_{21} & B_{22} & \cdots & B_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ B_{p1} & B_{p2} & \cdots & B_{pp} \end{bmatrix} \begin{pmatrix} \vec{T}_{-p+1} \\ \vec{T}_{-p+2} \\ \vdots \\ \vec{T}_1 \end{pmatrix} = \begin{pmatrix} \vec{F}^*_{-p+1} \\ \vec{F}^*_{-p+2} \\ \vdots \\ \vec{F}^*_1 \end{pmatrix} \tag{54}$$

where

$$B_{ij}=\int_0^{\Delta t} N_j a_j\Big(\mathbf{C}\dot{N}_i+\mathbf{K}N_i\Big)d\bar{\tau}$$

$$\vec{F}^*_j=\begin{pmatrix} \int_0^{\Delta t} N_j f_{-p+1,j}d\bar{\tau} \\ \int_0^{\Delta t} N_j f_{-p+2,j}d\bar{\tau} \\ \vdots \\ \int_0^{\Delta t} N_j f_{1,j}d\bar{\tau} \end{pmatrix}$$

$$i,j \in -p+1, -p+2, \cdots, 1$$

In the remainder of this section, for illustration, the relationships emanating from the finite element in time philosophy to the equivalent multi-step algorithms is described. Consider further relaxing the weak form by summing the representations given in (53) as follows:

$$\int_0^{\Delta t}\left(\sum_{-p+1}^{1} N_i a_i\right)\Big[\mathbf{C}\dot{\vec{T}}+\mathbf{K}\vec{T}-\vec{F}\Big]d\bar{\tau}=0 \tag{55}$$

As such, since we now have

$$W_{PG} = \sum_{-p+1}^{1} N_i a_i = w_0 + w_1\left(\frac{\bar{\tau}}{\Delta t}\right) + \cdots + w_p\left(\frac{\bar{\tau}}{\Delta t}\right)^p = W(\bar{\tau}) \tag{56}$$

where w_j's are function of a_i's, and

$$\vec{T} = \begin{bmatrix} N_{-p+1} & N_{-p+2} & \cdots & N_1 \end{bmatrix} \begin{pmatrix} \vec{T}_{n-p+1} \\ \vec{T}_{n-p+2} \\ \vdots \\ \vec{T}_{n+1} \end{pmatrix} = \sum_{i=-p+1}^{1} N_i(\bar{\tau})\vec{T}_{n+i} = \sum_{i=-p+1}^{1} \prod_{\substack{k=-p+1 \\ k\neq i}}^{1} \frac{\bar{\tau} - k\Delta t}{(i-k)\Delta t}\vec{T}_{n+i} \tag{57}$$

the relaxed weak form of representation, equation (55) becomes

$$\int_0^{\Delta t} W(\bar{\tau}) \left[\mathbf{C}\sum_{i=-p+1}^{1} \dot{N}_i(\bar{\tau})\vec{T}_{n+i} + \mathbf{K}\sum_{i=-p+1}^{1} N_i(\bar{\tau})\vec{T}_{n+i} - \sum_{i=-p+1}^{1} N_i(\bar{\tau})f_{n+i}\right] d\bar{\tau} = 0 \tag{58}$$

or

$$\left[\sum_{j=0}^{p-1}\sum_{i=-p+1}^{1}\int_0^{\Delta t} w_j \Delta t^j \dot{N}_i(\bar{\tau})\vec{T}_{n+i} d\bar{\tau}\right]\mathbf{C} + \left[\sum_{j=0}^{p-1}\sum_{i=-p+1}^{1}\int_0^{\Delta t} w_j \Delta t^j N_i(\bar{\tau})\vec{T}_{n+i} d\bar{\tau}\right]\mathbf{K} - \sum_{j=0}^{p-1}\sum_{i=-p+1}^{1}\int_0^{\Delta t} w_j \Delta t^j N_i(\bar{\tau}) f_{n+i} d\bar{\tau} = 0 \tag{59}$$

or

$$\left[\sum_{q=1}^{p}\frac{\Delta t^{q-1}\sum_{j=-p+1}^{1}N_j^{(q)}\vec{T}_{n+j}}{(q-1)!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i+1}\right]\mathbf{C} + \left[\sum_{q=0}^{p}\frac{\Delta t^{q-1}\sum_{j=-p+1}^{1}N_j^{(q)}\vec{T}_{n+j}}{q!}\sum_{i=0}^{p}\frac{w_i\Delta t^i}{q+i}\right]\mathbf{K} - \sum_{q=0}^{p}\int_0^{\Delta t} w_q\bar{\tau}^q\vec{F}_n d\bar{\tau} = 0 \quad (60)$$

Equation (60) which is a consequence of a class of finite element in time operators [FETInO] has the same representation as the multi-step representations [MSInO] given by equation (41), and is identical. In summary, a class of finite element in time operators were described within the framework of the present developments and are subsequently shown to have equivalence to the multi-step, or consequently the equivalent single-step approaches.

Concluding remarks

A formal theory of development/evolution with particular attention to characterization of a wide class of transient algorithms for heat transfer computations was described. Subsequently, an overview of recent developments describing the theoretical basis and the resulting consequences towards formalizing the fundamental concepts leading to a clear understanding of integral operators, a variety of single-step integration operators, multi-step integration operators, and a class of finite element in time integration operators was presented. Unlike previous efforts, although the developments emanate from a time weighted philosophy, the overall theoretical framework is based on new and different theoretical perspectives. Consequently, it therein serves as a prelude towards a clear understanding and an improved insight and provides a formal theory of development, evolution and characterization of time discretization operators. The formal relationships and equivalences amongst the various time operators was established. Different from the way the development of time discretized operators encompassing integral/integration operators have been described in traditional text books and in the research literature, the present developments provide a rich understanding of the theoretical basis and the fundamental principles. Finally, the discrete numerically assigned [DNA] algorithmic markers which comprise of both the weighted time fields and the conditions imposed upon the dependent field variable approximation serve well to uniquely characterize a wide class of time discretized operators. As such, in order to provide a standardized formal ideology when referring to and/or relating different algorithms to permit lucid communication, one may simply describe these in relation to the algorithmic DNA markers. A generalized implementation procedure for transient analysis was outlined to permit a single analysis code to incorporate a variety of features.

References

Argyris, J.H. and Scharpf, D.W. (1969), "Finite elements in time and space", *The Aeronautical Journal of the Royal Aeronautical Society*, Vol. 73, p. 1041.

Bathe, K.J. (1982), *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Englewood Cliffs, NJ.

Belytschko, T. and Hughes, T.J.R. (1983), *Computational Methods in Transient Analysis*, North Holland.

Chen, X., Tamma, K.K. and Sha, D. (1993), "A new virtual-pulse time integral methodology for linear transient heat transfer problems", *Numerical Heat Transfer, Part B – Fundamentals*, Vol. 24, p. 301.

Crank, J. and Nicolson, P. (1947), "A practical method for the numerical evaluation of solutions of partial differential equations of the heat conduction type", *Proc. Combridge Phil. Soc.*, Vol. 43, p. 50.

Dupont, T., Fairweather, G. and Johnson, P. (1974), "Three-level Galerkin methods for parabolic equations", *SIAMJ*, Vol. 11, p. 392.

Fried, I. (1969), "Finite element analysis of time-dependent phenomena", *AIAA J*, Vol. 7, p. 1170.

Gear, C.W. (1969), "The automatic integration of stiff ordinary differential equations", in Morrell, A.J.H. (Ed.), *Information Processing*, Vol. 68, North Holland, Dordrecht, p. 187.

Gurtin, M. (1964), "Variational principles for linear initial-value problems", *Q. Appl. Math*, Vol. 22, p. 252.

Lees, H. (1966), "A linear three-level differences scheme for quasilinear parabolic equations", *Mech. Comp.*, Vol. 20, p. 516.

Liniger, W. (1968), "Optimization of a numerical integration method for stiff systems of ordinary differential equations", *IBM Research Report*, No. RC2198.

Liniger, W. (1969), "Global accuracy stability of one and two step integration formulae for stiff ordinary differential equation", *Conf. Numerical Solution of Differential Equations*, Dundee University.

Manolis, G.D. and Beskos, D.E. (1980), "Thermally induced vibrations of beam structures", *Computer Methods in Applied Mechanics and Engineering*, Vol. 21 No. 337.

Mei, Y., Mohan, R.V. and Tamma, K.K. (1994), "Evaluation and applicability of a new explicit time integral methodology for transient thermal problems – finite volume formulations", *Numerical Heat Transfer*, Vol. 6 No. 3, p. 313.

Oden J.T. (1969), "A general theory of finite elements, part II", *International Journal for Numerical Methods in Engineering*, Vol. 1, p. 247.

Tamma, K.K. and Railkar, S.B. (1987a), "A generalized hybrid transfinite element computational approach for nonlinear/linear unified thermal/structural analysis", *Computers and Structures*, Vol. 26, p. 655.

Tamma, K.K. and Railkar, S.B. (1987b), "Nonlinear/linear unified thermal stress formulations: transfinite element approach", *Comp. Meths. Appl. Mech. Eng.*, Vol. 64, p. 415.

Tamma, K.K., Chen, X. and Sha, D. (1994), "Further developments towards a new virtual-pulse time integral methodology for general non-linear transient thermal analysis", *Communications in Applied Numerical Methods in Engineering*, Vol. 10, p. 961.

Tamma, K.K., Mei, Y., Chen, X. and Sha, D. (1995), "Recent advances towards an effective virtual-pulse (VIP) explicit time integral methodology for multidimensional thermal analysis", *International Journal of Numerical Methods in Fluids*, Vol. 20, p. 523.

Tamma, K.K., Zhou, X. and Sha, D. (1998), "On a plausible standardized and formal theory of development/evolution of a wide class of computational structural dynamics algorithms", in *AIAA SDM Conf.*, AIAA-98-2016, Long Beach, CA.

Tamma, K.K., Zhou, X. and Sha, D. (1999), "Transient algorithms for heat transfer: general developments and approaches for generating *N*th-order time accurate operators including practically usefull second-order forms", *International Journal for Numerical Methods in Engineering* Vol. 44, p. 1545..

Tamma, K.K., Zhou, X. and Valasutean, R. (1997), "Computational algorithms for transient analysis: the burden of weight and consequences towards formalizing discrete numerically assigned (DNA) algorithmic markers: Wp-family", *Computer Methods in Applied Mechanics and Engineering*, Vol. 149, p. 153.

Washizu, K. (1975), *Variational Methods in Elasticity and Plasticity*, 2nd ed., Pergamon Press, Oxford.

Wood, W.L. (1987), *Numerical Methods for Transient and Coupled Problems*, Chapter 8, John Wiley and Sons Ltd, p. 149.

Wood, W.L. (1990), *Practical Time Stepping Schemes*, Clarendon Press, Oxford.

Zienkiewicz, O.C. and Taylor, R.L. (1994), *The Finite Element Method*, Vol. 1, McGraw-Hill, New York, NY.

Zlamal, M. (1977), *Finite Elements Methods in Heat Condution Problems*, Academic Press, New York, NY.